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Morita-extensions and nearness-completions

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Abstract

In the context of nearness spaces, Morita's 1951 concepts of simple extensions, completeness, and completions are investigated and compared with the corresponding concepts in current use.

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Dedicated to the memory of Kiiti Morita

Introduction

In 1951 Morita [13] published the groundbreaking papers "On the simple extension of a space with respect to a uniformity I–IV". In these papers—of which A.K. Steiner and E.F. Steiner [19] in 1973 would say "Morita's paper is little known and consequently his original ideas are not referred to as often as they should be"—he demonstrated that strict extensions of spaces can be conveniently obtained and studied by means of suitable uniform-like structures via a completion process. This idea was to play a key role in further investigations of strict topological extensions. Although the structures used by Smirnov [17,18] for Hausdorff compactifications, respectively by Ivanova and Ivanov [12] and Terwilliger [20] for strict T_1-compactification (namely, proximities, respectively contiguities) appear to be of a different kind, they fit—as a closer inspection shows—perfectly into Morita's framework.

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Later Rinow [15] and Harris [8] independently returned to Morita's ideas and improved the theory by suitably modifying Morita's concept of completeness. Unfortunately, however, in order to achieve maximal generality these authors did not require the systems of (uniform) covers under consideration to satisfy suitably selected axioms. A disadvantage of their approach (compared, e.g., to that of uniform structures) is that it forces the authors to consider different covering systems to be equivalent. Thus their structures are equivalence classes of covering systems. This is the price the authors have to pay for the generality obtained by the lack of suitable axioms. Moreover, this price has to be paid not only once but causes further heavy losses in the development of the theory. Such a bargain strikes us as unsatisfactory. In 1973 A.K. Steiner and E.F. Steiner [19] chose a conceptually favourable and far more elegant approach by replacing the star-refinement axiom in Tukey's axiomatization of uniform spaces by a slightly localized version. Their theory of *semi-uniform spaces* is today the most elegant variant of Morita's theory for the study of regular extensions of topological spaces. Its only weakness is its restriction to the regular case. This deficiency was finally removed by Herrlich's 1974 concept of *nearness spaces*, [1] obtained by replacing Tukey's star-refinement axiom by a strongly localized version. We feel that the theory of nearness spaces provides a mature and perhaps final variant of Morita's original theory. In fact, in his 1989-paper "Extensions of mappings I" [14] Morita himself completely adopted the conceptual framework of the theory of nearness spaces, unfortunately, however, not the corresponding terminology.

The purpose of this note is to compare Morita's original concepts with the corresponding nearness concepts.

1. Terminology

We use the familiar nearness terminology (see, in particular, [4] or [11]). To facilitate comparison with the terminology used by Morita in 1951 [13], respectively 1989 [14], a small dictionary follows in Table 1. All spaces are supposed to be T_1-spaces.

2. Morita-completeness versus completeness

A nearness space X is called

- *Morita-complete* provided in X every strong Cauchy filter converges,
- *complete* provided in X every cluster has an adherence point (equivalently: every round Cauchy filter converges).

Proposition 2.1 [13,19,9]. *A regular space is Morita-complete iff it is complete.*

[1] The name *nearness space* is due to the fact that Herrlich's original axiomatization [9] did not use the concept of *uniform covers* as basic, but the concept of collections being *near*. That these two concepts (as well as Katětov's concept of *micromeric* collections) properly axiomatized lead to isomorphic concrete categories was observed in [10].

Table 1

nearness term	Morita [14]	Morita [13]
nearness space	generalized uniform space	T-uniform space
separated nearness space	—	—
regular nearness space	semi-uniform space	regular T-uniform space
uniform space	uniform space	completely regular T-uniform space
complete	complete	—
—[a]	—	complete
Cauchy filter	Cauchy filter	—
strong Cauchy filter	strict Cauchy filter	Cauchy family
round Cauchy filter	weak star-filter	—
minimal Cauchy filter	minimal Cauchy filter	—
cluster	—	—

[a]In this paper we will use the term *Morita-complete* for the above entry. Other kinds of completeness, investigated, e.g., by Harris [8], Rinow [15,16], and Carlson [6,7] will not be discussed here. For further historical notes see [5].

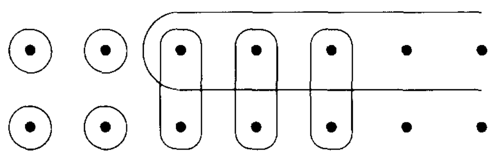

Fig. 1.

Proposition 2.2. *Every complete nearness space is Morita-complete.*

Proof. Let X be complete and let $\mathcal{F}$ be a strong Cauchy filter on X. Then the collection $\mathfrak{G}$ of all sets that are uniform neighbourhoods of some member of $\mathcal{F}$ is a round Cauchy filter (equivalently: the collection $\sec \mathfrak{G}$ of all subsets of X that meet every member of $\mathfrak{G}$ is a cluster). Thus $\mathfrak{G}$ and, consequently, $\mathcal{F}$ converge. □

Example 2.3. A Morita-complete, separated nearness space that fails to be complete.

Consider $X = \mathbb{N} \times \{1, 2\}$. Let a cover $\mathfrak{U}$ of X be uniform provided that the following conditions are satisfied:

$$\exists U \in \mathfrak{U}\ \exists n \in \mathbb{N}\ \forall m \geqslant n\ (m, 2) \in U, \tag{1}$$

$$\exists n \in \mathbb{N}\ \forall m \geqslant n\ \exists U \in \mathfrak{U}\ \{(m, 1), (m, 2)\} \subset U. \tag{2}$$

The associated nearness space (see Fig. 1) has the prescribed properties (cf. [4, Example 3.2]).

Proposition 2.4.

(1) *A regular nearness space is (Morita-) complete iff it has no proper dense regular extension.*

(2) *A separated nearness space is complete iff it has no proper dense separated extension.*

Proof. (1) See [4, Theorem 2.1] or [11, Theorem 7.3.4].

(2) See [4, Theorem 3.4] or [11, Theorem 7.3.3]. □

Example 2.5. A Morita-complete separated nearness space with a proper dense separated extension.

Consider the nearness space of Example 2.3. It has the prescribed properties since its completion is a proper dense separated extension.

Proposition 2.6.

(1) *(Morita-) complete regular nearness spaces form an epireflective subcategory of the category of regular nearness spaces.*

(2) *Complete separated nearness spaces form an epireflective subcategory of the category of separated nearness spaces.*

(3) *Morita-complete separated nearness spaces do not form a reflective subcategory of the category of nearness spaces.*

Proof. (1) See [13, II, Theorem 3] or [19, Theorem 3.5] or [9, Theorem 8.12].

(2) See [3, Corollary 2.5].

(3) Reflective subcategories are closed under the formation of limits. Example 2.7, however, shows that there exists a Morita-complete separated nearness space X that has Morita-complete subspaces X_1 and X_2 such that $X_1 \cap X_2$ fails to be Morita-complete. □

Example 2.7. Consider the following Morita-complete separated nearness space (see Fig. 2) $X = \mathbb{N} \times \{0, 1, 2\}$. A cover $\mathfrak{U}$ of X is uniform provided that the following conditions are satisfied:

$$\exists U \in \mathfrak{U}\ \exists n \in \mathbb{N}\ \forall m \geqslant n\ (m, 1) \in U, \tag{3}$$

$$\exists n \in \mathbb{N}\ \forall m \geqslant n\ \exists U \in \mathfrak{U}\ \{(m,0),(m,1)\} \subset U, \tag{4}$$

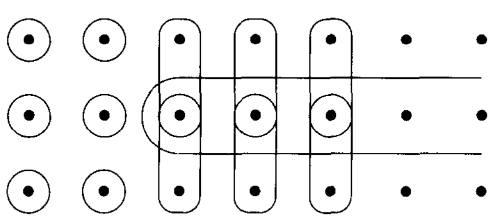

Fig. 2.

$$\exists n \in \mathbb{N}\ \forall m \geqslant n\ \exists U \in \mathfrak{U}\ \{(m,1),(m,2)\} \subset U. \tag{5}$$

Then X and the subspaces determined by the sets $X_1 = \mathbb{N} \times \{0,1\}$ and $X_2 = \mathbb{N} \times \{1,2\}$ are Morita-complete and separated, but the subspace determined by $X_1 \cap X_2 = \mathbb{N} \times \{1\}$ fails to be Morita-complete.

3. Simple extensions and Morita-completions versus completions

Morita [13] constructed for each nearness space X an extension X_{M}, which he called the *simple extension* of X, and—via transfinite iteration of this construction—a Morita-complete extension X_{M}^* of X, which we will call the *Morita-completion* of X. Later Herrlich [9] constructed for each nearness space X a *completion* X^*. For regular X, the simple extension X_{M} is Morita-complete [13, I, Theorem 9] and thus X_{M} and X_{M}^* coincide. Moreover, since in the regular case X_{M}^* (see [13, II, Theorem 3] or [19, Theorem 3.5]) as well as X^* (see [9, Theorem 8.12]) form complete regular reflections of X, the Morita-completion of X and the completion of X are equivalent extensions of X. For separable nearness spaces X the extensions X_{M}^* and X^* may fail to be equivalent as Example 2.3 demonstrates. That in general X_{M} may fail to be Morita-complete has been shown by Morita himself (see [13, III, Example on p. 168]). Here we give another, perhaps simpler, example of a separated nearness space X with $X_{\mathrm{M}} \neq X_{\mathrm{M}}^*$.

Example 3.1. Consider the following nearness space: $Z = (\mathbb{N}^2 \times \{0,1\}) \cup \mathbb{N} \cup \{\infty\}$. A cover $\mathfrak{U}$ of Z is uniform provided that it satisfies the following conditions:

$$\forall n \in \mathbb{N}\ \exists m \in \mathbb{N}\ \exists U \in \mathfrak{U}\ \big(\{n\} \cup \{(k,n,0) \mid k \geqslant m\}\big) \subset U, \tag{6}$$

$$\exists n \in \mathbb{N}\ \exists U \in \mathfrak{U}$$
$$\big(\{\infty\} \cup \{m \in \mathbb{N} \mid m \geqslant n\} \cup \{(k,l,0) \mid k \geqslant n \text{ and } l \geqslant n\}\big) \subset U, \tag{7}$$

$$\exists n \in \mathbb{N}\ \forall k \geqslant n\ \forall l \geqslant n\ \exists U \in \mathfrak{U}\ \{(k,l,0),(k,l,1)\} \subset U. \tag{8}$$

Then Z is a separated nearness space, and so are its nearness subspaces Y and X determined by the sets $Y = Z \backslash \{\infty\}$ and $X = Y \backslash \mathbb{N}$. Moreover:

(a) X is not Morita-complete.
(b) Y is (up to isomorphism) the simple extension X_{M} of X.
(c) Y is not Morita-complete.
(d) Z is (up to isomorphism) the simple extension Y_{M} of Y, the Morita-completion X_{M}^* of X and the completion X^* of X.

Thus in this case the completion and the Morita-completion of X coincide (up to isomorphism), but whereas the completion (as always) is obtained in one step, the Morita-completion requires two steps.

Proposition 3.2. *For any nearness space X,*

(1) *the simple extension X_{M} of X can be regarded in a canonical way as a subspace of the completion X^* of X,*

(2) *the Morita-completion X^*_{M} of X can be regarded in a canonical way as a subspace of the completion X^* of X.*

Proof. (2) follows from (1) in view of the fact (see [1]) that whenever X is a nearness subspace of Y and Y is a nearness subspace of X^*, then Y^* is canonically isomorphic to X^*. (1) is due to Rinow (see [15, Theorem 4.2]). Since Rinow's terminology differs from ours we include a proof:

Let X be a nearness space. With every strong Cauchy filter $\mathcal{F}$ we associate the round Cauchy filter $\mathfrak{G}(\mathcal{F})$ consisting of all sets that form a uniform neighbourhood of some member of $\mathcal{F}$, and the cluster $f(\mathcal{F}) = \sec \mathfrak{G}(\mathcal{F})$ consisting of all subsets of X that meet each member of $\mathfrak{G}(\mathcal{F})$. If strong Cauchy filters $\mathcal{F}$ and $\mathcal{F}'$ are Morita-equivalent, then $\mathfrak{G}(\mathcal{F}) = \mathfrak{G}(\mathcal{F}')$, and thus $f(\mathcal{F}) = f(\mathcal{F}')$. Hence the correspondence f induces a unique map $f\colon X_{\mathrm{M}} \to X^*$. If $f(\mathcal{F}) = f(\mathcal{F}')$, then

$$\mathfrak{G}(\mathcal{F}) = \sec^2\bigl(\mathfrak{G}(\mathcal{F})\bigr) = \sec\bigl(f(\mathcal{F})\bigr) = \sec\bigl(f(\mathcal{F}')\bigr) = \sec^2\bigl(\mathfrak{G}(\mathcal{F}')\bigr) = \mathfrak{G}(\mathcal{F}').$$

Thus f is injective.

Let G be an open set in X and let G_{M} respectively $\widetilde{G}$ be the largest open subset C of X_{M} respectively of X^* with $X \cap C = G$. Let $\mathcal{F}$ be a strong Cauchy filter on X and let $p_{\mathcal{F}}$ respectively $p_{f(\mathcal{F})}$ be the point of X_{M} respectively X^* that corresponds to $\mathcal{F}$ respectively $f(\mathcal{F})$ (in the canonical construction of X_{M} respectively X^*). It suffices to show that $p_{\mathcal{F}} \in G_{\mathrm{M}}$ iff $p_{f(\mathcal{F})} \in \widetilde{G}$, since this implies (by strictness) that $f[X_{\mathrm{M}}]$ is a subspace of X^*. Thus the equivalences

$$\begin{aligned} p_{\mathcal{F}} \in G_{\mathrm{M}} &\Leftrightarrow G \in \mathfrak{G}(\mathcal{F}) \Leftrightarrow (X\backslash G) \notin f(\mathcal{F}) \Leftrightarrow p_{f(\mathcal{F})} \notin \mathrm{cl}_{X^*}(X\backslash G) \\ &\Leftrightarrow p_{f(\mathcal{F})} \in X^*\backslash \mathrm{cl}_{X^*}(X\backslash G) \Leftrightarrow p_{f(\mathcal{F})} \in \widetilde{G} \end{aligned}$$

complete the proof. □

Finally we provide an example of a dense embedding $X \hookrightarrow Y$ of a Morita-complete nearness space X into a separated nearness space Y that fails to be Morita-complete. Thus, loosely speaking, the addition of accumulation points to a space may create a hole in the space:

Example 3.3. Consider the following nearness space: $Y = (\mathbb{N}^2 \times \{0,1\}) \cup \mathbb{N}$. A cover $\mathfrak{U}$ of Y is uniform provided that it satisfies the following conditions:

$$\forall n \in \mathbb{N}\ \exists m \in \mathbb{N}\ \exists U \in \mathfrak{U}\ \bigl(\{n\} \cup \bigl\{(k,n,0) \mid k \geqslant m\bigr\}\bigr) \subset U, \tag{9}$$

$$\exists n \in \mathbb{N}\ \exists U \in \mathfrak{U}\ \bigl(\bigl\{m \in \mathbb{N} \mid m \geqslant n\bigr\} \cup \bigl\{(k,l,0) \mid k \geqslant n \text{ and } l \geqslant n\bigr\}\bigr) \subset U, \tag{10}$$

$$\exists n \in \mathbb{N}\ \forall k \geqslant n\ \forall l \in \mathbb{N}\ \exists U \in \mathfrak{U}\ \bigl\{(k,l,0),(k,l,1)\bigr\} \subset U. \tag{11}$$

Then Y is a separated nearness space, and so is its dense nearness subspace X determined by the set $X = Y\backslash\mathbb{N}$. Whereas X is Morita-complete, Y fails to be so.

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